Chaos in generalized Jaynes-Cummings model. Kinetic approach

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Abstract

In this work we study possibility of chaos formation in the dynamics governed by paradigmatic model of Cavity Quantum Electrodynamics, the so called James-Cammings model. In particular we consider generalized JC model. It is shown that even in the case of zero detuning dynamics is chaotic. Kinetic approach for the problem under study has been applied.
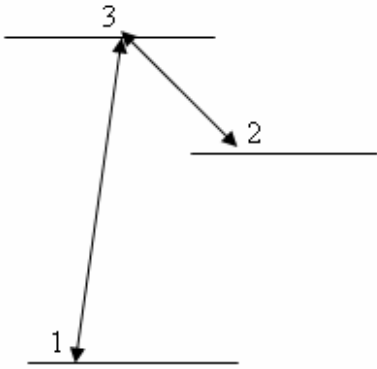
FIG. 1: Schematic representation of atomic levels and available inter level transitions for generalized JC model.

I.   INTRODUCTION

Cavity quantum electrodynamics (CQED) is a rapidly developing field of physics studying the interaction of atoms with photons in the high-finesse cavities [1, 2, 3, 4]. Interest to such a systems basically is caused by two facts: One of them is the possibility of more deep understanding of quantum dynamics of open systems. Second argument is the possibility of practical application in the field of quantum computing [5].

In particular CQED experiments implement a situation so simple that their results are of great importance for better understanding of fundamental postulates of quantum theory [6]. They are thus appropriate for tests of basic quantum properties: quantum superposition, complementarily or entanglement. In the context of quantum information processing, the atom and cavity are long-lived qubits, and their mutual interaction provides a controllable entanglement mechanism an essential requirement for quantum computing [6, 7].

In general dissipation processes must be taken into account when discussing problems of CQED. In particular there are two dissipative channels for system: the atom may spontaneously emit into modes other then preferred cavity mode, and photons may pass through the cavity output coupling mirror.

But modern experiments in CQED have achieved strong atom-field coupling for the strength of the coupling exceeds both decay processes [8, 9, 10]. If so, then problem is reduced to the driven Jaynes-Cummings (JC) Hamiltonian, which models the interaction of a single mode of an optical cavity having resonant frequency with a two level atom comprised

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of a ground and exited states [11].

This model is basic model of interaction of radiation with matter and describes the energy exchange between atom and quantized radiation field in an ideal lossless cavity.

In further was offered JC model generalized for three level optical atoms (see Fig.1). For more details see [12, 13].

In most general case atom-radiation field interaction should involve not only the internal atomic transitions and field states but also the center-of-mass motion of the atom. With the recoil effect taken into account, generalized JC Hamiltonian takes form

$$\hat{H} = \frac{\hat{p}^2}{2m} + \sum_{\alpha=1}^{2} \hbar\omega_\alpha \hat{b}_\alpha^+ \hat{b}_\alpha + \sum_{j=1}^{3} E_j \hat{R}_{jj} + \hbar \sum_{\alpha=1}^{2} \hat{g}_\alpha (\hat{b}_\alpha \hat{R}_{3\alpha} + \hat{b}_\alpha^+ \hat{R}_{\alpha3}) \cos(x k_{f_\alpha}) \qquad (1)$$

here $\hat{x}$ and $\hat{p}$ are coordinate and pulse of the atom $R_{3\alpha}$, $R_{\alpha3}$ are transition operators $\alpha = 1, 2$, $\hat{R}_{jj}$ is the operator of the level population, satisfying the condition $\sum_{j=1}^{3} R_{jj} = 1$. In the basis of atomic states

$$|a_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, |a_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |a_3\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, R_{jj} \text{ operators are the generators of the } SU(3)$$

$$\text{group } |\hat{R}_{11}\rangle = \begin{pmatrix} 0\,0\,0 \\ 0\,0\,0 \\ 0\,0\,1 \end{pmatrix}, |\hat{R}_{12}\rangle = \begin{pmatrix} 0\,0\,0 \\ 0\,0\,0 \\ 0\,1\,0 \end{pmatrix}, |\hat{R}_{13}\rangle = \begin{pmatrix} 0\,0\,0 \\ 0\,0\,0 \\ 1\,0\,0 \end{pmatrix}, |\hat{R}_{21}\rangle = \begin{pmatrix} 0\,0\,0 \\ 0\,0\,1 \\ 0\,0\,0 \end{pmatrix},$$

$$|\hat{R}_{22}\rangle = \begin{pmatrix} 0\,0\,0 \\ 0\,1\,0 \\ 0\,0\,0 \end{pmatrix}, |\hat{R}_{23}\rangle = \begin{pmatrix} 0\,0\,0 \\ 1\,0\,0 \\ 0\,0\,0 \end{pmatrix}, |\hat{R}_{31}\rangle = \begin{pmatrix} 0\,0\,1 \\ 0\,0\,0 \\ 0\,0\,0 \end{pmatrix}, |\hat{R}_{32}\rangle = \begin{pmatrix} 0\,1\,0 \\ 0\,0\,0 \\ 0\,0\,0 \end{pmatrix},$$

$$|\hat{R}_{33}\rangle = \begin{pmatrix} 1\,0\,0 \\ 0\,0\,0 \\ 0\,0\,0 \end{pmatrix}.$$

Model given by the Hamiltonian (1) is nonlinear and non-integrable and due to this may display chaotic behavior. Analogous problem but for ordinary two level JC model was studied in [14] in detail. Result obtained in [14] is that dynamics essentially depends on detuning parameter between inter level transition frequency and frequency of radiation field $\delta = \omega_\alpha - E/\hbar$. Namely, in case of zero detuning system displays regular behavior. Goal of this work is the study of nonlinear dynamics governed by Hamiltonian (1). For

better understanding of the problem we shall discuss resonant case. We shall try to find out difference in dynamics between generalized and ordinary JC model studied in [14].

In resonant case, complete set of Heisenberg equations of motion corresponding to the Hamiltonian (1) looks like

$$
\begin{aligned}
\frac{dx(\tau)}{d\tau} &= \alpha \; p(\tau), \\
\frac{dp(\tau)}{d\tau} &= u_1(\tau)\sin(x) + \Omega \; ku_2(\tau)\sin(kx), \\
\frac{dR_1(\tau)}{d\tau} &= \nu_1(\tau)\cos(x), \\
\frac{dR_2(\tau)}{d\tau} &= \Omega \; \nu_2(\tau)\cos(kx), \\
\frac{d\nu_1(\tau)}{d\tau} &= 2\cos(x)(M_1+1)(1-2R_1(\tau)-R_2(\tau)) - \Omega\cos(kx)\cdot B(\tau), \\
\frac{d\nu_2(\tau)}{d\tau} &= 2\Omega\cos(kx)(M_2+1)(1-R_1(\tau)-2R_2(\tau)) - \cos(x)\cdot B(\tau), \\
\frac{du_1(\tau)}{d\tau} &= \Omega\cos(kx)\cdot C(\tau), \\
\frac{du_2(\tau)}{d\tau} &= -\cos(kx)\cdot C(\tau), \\
\frac{dB(\tau)}{d\tau} &= \cos(kx)(M_1+1)\nu_2(\tau) + \Omega\cos(kx)(M_2+1)\nu_1(\tau), \\
\frac{dC(\tau)}{d\tau} &= \cos(x)(M_1+1)u_2(\tau) - \Omega\cos(kx)(M_2+1)u_1(\tau) \\
\frac{dN_1(\tau)}{d\tau} &= \cos(x)\cdot\;\nu_1(\tau), \\
\frac{dN_2(\tau)}{d\tau} &= \Omega\cos(kx)\cdot\;\nu_2(\tau)
\end{aligned}
\tag{2}
$$

In (2) transform to the Hermit variables $\hat{A}_1 = \imath(\hat{b}_1\hat{R}_{31} - \hat{b}_1^+\hat{R}_{13})$, $\hat{A}_2 = \imath(\hat{b}_2\hat{R}_{32} - \hat{b}_2^+\hat{R}_{23})$, $\hat{U}_1 = (\hat{b}_1\hat{R}_{31} + \hat{b}_1^+\hat{R}_{13})$, $\hat{U}_2 = (\hat{b}_2\hat{R}_{32} + \hat{b}_2^+\hat{R}_{23})$, $\hat{B} = \hat{b}_1\hat{b}_2^+\hat{R}_{21} + \hat{b}_1^+\hat{b}_2\hat{R}_{12}$, $\hat{C} = \imath(\hat{b}_1\hat{b}_2^+\hat{R}_{21} - \hat{b}_1^+\hat{b}_2\hat{R}_{12})$, $\hat{N}_1 = \hat{b}_1^+\hat{b}_1$, $\hat{N}_2 = \hat{b}_2^+\hat{b}_2$,

and procedure of semi-classical averaging is done [14] $x = k_{f_1} < \hat{x} >$, $\quad p = \frac{<\hat{P}>}{\hbar k_{f_1}}$, $R_1 = < R_{11} >$, $R_2 = < R_{22} >$, $\nu_1 = < \hat{A}_1 >$, $\nu_2 = < \hat{A}_2 >$, $u_1 = \left\langle \hat{U}_1 \right\rangle$, $u_2 = \left\langle \hat{U}_2 \right\rangle$, $k = \frac{k_{f_2}}{k_{f_1}}$, $B = \left\langle \hat{B} \right\rangle$, $C = \left\langle \hat{C} \right\rangle$, $\quad \tau = g_1 t$, $\quad \alpha = \frac{k_{f_1}^2\hbar}{g_1 m}$, $\quad \Omega = \frac{g_2}{g_1}$. In addition following notations for motion integrals are introduced $M_1 = \hat{N}_1 - \hat{R}_{11} = const$, $M_2 = \hat{N}_2 - \hat{R}_{22} = const$.

Below the set of equations (2) will be the object of our interest. We shall consider two cases:

1) Variable x is slow as against $u_1(\tau)$, $u_2(\tau)$. In this case set of equations (2) separates into two subsystems

$$\frac{du_1(\tau)}{d\tau} = \Omega \cos(kx) \cdot C(\tau)$$

$$\frac{du_2(\tau)}{d\tau} = -\cos(x) \cdot C(\tau),$$

$$\frac{dC(\tau)}{d\tau} = \cos(x)(M_1 + 1)u_2(\tau) - \Omega \cos(kx)(M_2 + 1)u_1(\tau), \qquad (3)$$

$$\frac{dx(\tau)}{d\tau} = \alpha p(\tau),$$

$$\frac{dp(\tau)}{d\tau} = u_1(\tau)\sin(x) + \Omega k u_2(\tau)\sin(kx).$$

and

$$\frac{dR_1(\tau)}{d\tau} = \nu_1(\tau)\cos(x),$$

$$\frac{dR_2(\tau)}{d\tau} = \Omega \nu_2(\tau)\cos(kx),$$

$$\frac{d\nu_1(\tau)}{d\tau} = 2\cos(x)(M_1 + 1)(1 - 2R_1(\tau) - R_2(\tau)) - \Omega \cos(kx)B(\tau),$$

$$\frac{d\nu_2(\tau)}{d\tau} = 2\Omega \cos(kx)(M_2 + 1)(1 - R_1(\tau) - 2R_2(\tau)) - \cos(kx)B(\tau), \qquad (4)$$

$$\frac{dB(\tau)}{d\tau} = \cos(x)(M_1 + 1)\nu_2(\tau) + \Omega \cos(kx)(M_2 + 1)\nu_1(\tau),$$

$$\frac{dN_1(\tau)}{d\tau} = \cos(x) \cdot \nu_1(\tau),$$

$$\frac{dN_2(\tau)}{d\tau} = \Omega \cos(kx) \cdot \nu_2(\tau),$$

After solving of first three equations of (3), we have

$$C(\tau) = A\sin(\Omega, \tau),$$

$$u_1(\tau) = u_1(0) - \frac{A\Omega}{\Omega_1}\cos(\Omega_1 \tau), \qquad (5)$$

$$u_2(\tau) = u_2(0) + \frac{A\Omega}{\Omega_1}\cos(\Omega_1 \tau),$$

where

$$\Omega_1^2 = \cos^2(x)(M_1 + 1) + \Omega^2 \cos^2(kx)(M_2 + 1).$$

Then if, $k = \frac{k_{f_1}}{k_{f_2}} = 1$, motion of the atom inside of quantum cavity satisfies the equation

$$\frac{d^2x}{d\tau^2} + |\alpha|(u_1(0) + u_2(0))\sin(x) = 0, \qquad (6)$$

with the possible solutions

$$\frac{dx}{d\tau} = p = 2\Theta\omega_0 \begin{cases} cn(\tau, \Theta), & \Theta \leqslant 1 \\ dn(\tau, 1/\Theta), & \Theta \geqslant 1 \end{cases}, \qquad (7)$$

where $\Theta = \frac{1}{2}\left(1 + H/\omega_0^2\right)$, $\omega_0^2 = |\alpha|(u_1(0)+u_2(0))$, $H = \frac{p^2(0)}{2}+\omega_0^2\cos(x_0)$ and $cn(...)$, $dn(...)$ are Jacobi elliptic functions [15]. Two solutions (7) correspond to the two different phase trajectories. One of those solutions $dn(...)$ corresponds to the closed phase trajectory. Other one corresponds to open phase trajectory. In case of closed phase trajectory it means that

after the time interval , atom iterates initial state. For solving of the subsystem (4) we can use the method given in [13]. As a result we get:

$$R_1(\tau) = \mu(\cos(\lambda\tau) - 1) + \beta\sin(\lambda\tau) + \lambda_1^2[u(\cos(2\lambda\tau) - 1) + \nu\sin(2\lambda\tau)] + R_1(0),$$
$$R_2(\tau) = -\mu(\cos(\lambda\tau) - 1) - \beta\sin(\lambda\tau) + \lambda_2^2[u(\cos(2\lambda\tau) - 1) + \nu\sin(2\lambda\tau)] + R_2(0) \tag{8}$$

where

$$\mu = \lambda^{-4}[\lambda^2[\lambda_2^2 R_1(0) - \lambda_1^2 R_2(0)] + (\lambda_2^2 - \lambda_1^2)K], \quad \lambda = \sqrt{\lambda_1^2 + \lambda_2^2},$$

$$\beta = \lambda^{-3}[\lambda_2^2 \dot{R}_1(0) - \lambda_1^2 \dot{R}_2(0)], \quad u = \frac{1}{2}\lambda^{-4}[\lambda^2[2R_1(0) + 2R_2(0) - 1] + K],$$

$$\nu = \frac{1}{2}\lambda^{-3}[\dot{R}_1(0) + \dot{R}_2(0)], \quad \lambda_1 = \cos(x)\sqrt{M_1 + 1}, \quad \lambda_2 = \Omega\cos(kx)\sqrt{M_2 + 1},$$

and

$$N_1(\tau) = \mu(\cos(\lambda\tau) - 1) + \beta\sin(\lambda\tau) - \lambda_1^2[u(\cos(2\lambda\tau) - 1) + \nu\sin(2\lambda\tau)] + N_1(0),$$

$$N_2(\tau) = -\mu(\cos(\lambda\tau) - 1) - \beta\sin(\lambda\tau) + \lambda_2^2[u(\cos(2\lambda\tau) - 1) + \nu\sin(2\lambda\tau)] + N_2(0).$$

Really the quantities $R_1(\tau)$, $R_2(\tau)$, $R_3(\tau) = 1 - R_1(\tau) - R_2(\tau)$ are observable on the experiment. But in general case, when variable $x(\tau)$ is not adiabatic, we have to investigate (2) by use of numerical methods. If $x(\tau)$ is non-adiabatic variable, origination of chaos in the system is quite possible. In this case dynamical consideration loses sense and transition from dynamic to the statistical description is needed. As the result, statistical conceptions, namely Kolmogorov entropy and fractional dimension becomes important. Those concepts being statistical are strongly related to the systems motional characteristics, namely to the local instability of phase trajectories [16, 17, 18]. Results of numeric calculations, for the realistic values of parameters corresponding to the high-finesse Fabry-Perot cavity and real atoms [8, 10] are presented on Fig.2, Fig.3.

For determination of the width of Furrier transform of the correlation function we used method of fast Furrier transform

$$G_x(\tau) = <x(t+\tau)x(t)>$$

$$G_x(\omega) = \int_0^\infty d\tau G_x(\tau)\exp(\imath\omega\tau) = \frac{\tau_c}{1 + \omega^2\tau_c^2}$$
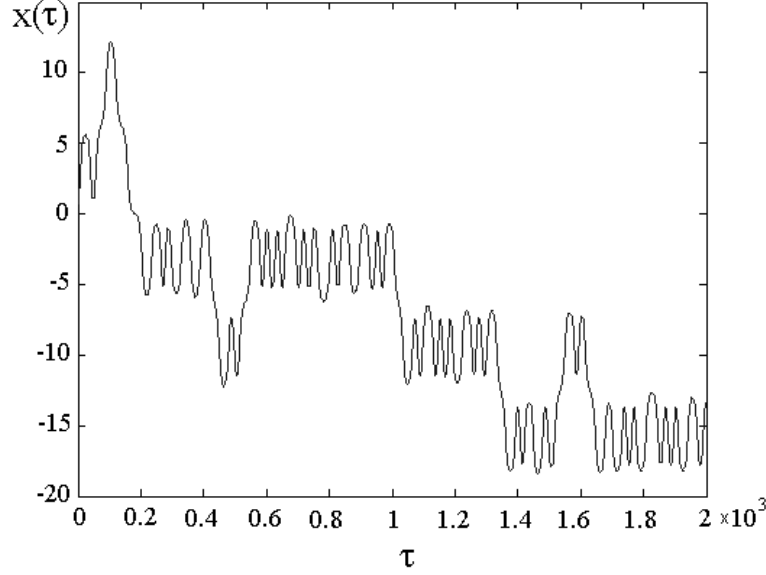
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FIG. 2: Plot of atom mass center coordinate dependence on time. As one can see from this Fig. motion is chaotic. As every other numeric result, plot is obtained for the values of parameters $\alpha = 0.01,\ \Omega = 0.5,\ k=3,\ M_1 = 1.6,\ M_2 = 1.7$

where $< (...) >= \lim\limits_{T \to \infty} \frac{1}{T} \int\limits_0^T (...)dt$ means time average, τ_c is the correlation time. Result of numeric calculations are presented on Fig.3. Nonzero width of the Fourier form is the sing of chaos.

One more sign for confirmation of chaos existence is the values of maximal Lyapunov exponent. In this sense it is worth to compare our result with the result obtained for ordinary JC model [14]. For numerical calculation of maximal Lyapunov exponent we shall use G. Benettin's algorithm consisting in the following procedures[18]: Solving two different systems starting from the initial points $x_0,\ x_0 + \bar{x}_0^0,\ |\bar{x}_0^0| = \varepsilon$, where ε is small initial distance between phase trajectories. Then after each step $x(t_i) = x_i,\ \bar{x}(t_i) = x_i$ one has to do re-scaling $\bar{x}_i^0 = \varepsilon \bar{x}_i/|\bar{x}_i|$ use as initial conditions $x_i, x_i + \bar{x}_i^0$, and so on up to the $x_N, x_N + \bar{x}_N^0$. Then Lyapunov exponent is given via

$$\Lambda = \frac{1}{N} \sum_{k=1}^{\mathbb{N}} \ln \left(\frac{\bar{x}_k}{\varepsilon} \right). \tag{9}$$

Result of numeric calculations for the maximal Lyapunov exponent is presented on Fig.4.

It is evident from the Fig.4, that in case of ordinary JC model corresponding to the values
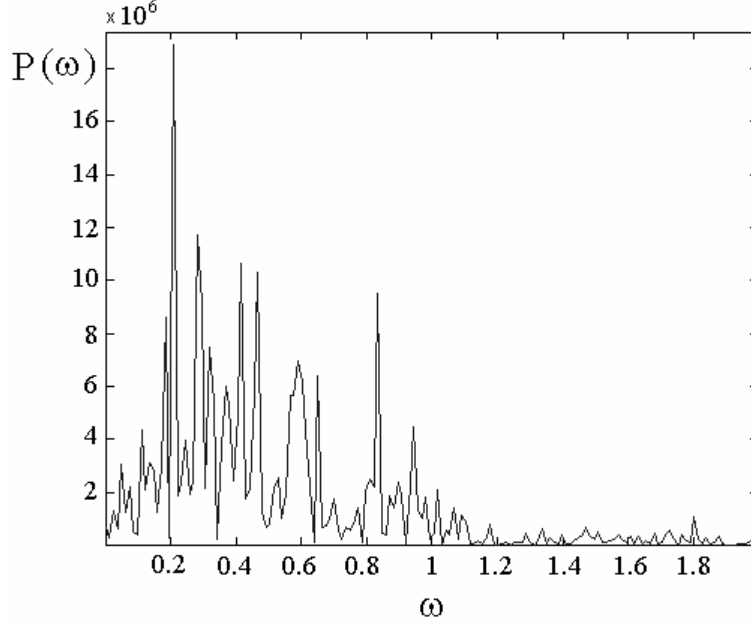
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FIG. 3: Fourier transform of the correlation function $p(\omega) = G_p(\omega) = \int_0^\infty d\tau \exp(\imath\omega\tau)G_p(\tau)$. Nonzero width of the correlation function is the sign of classical dynamical stochastisity
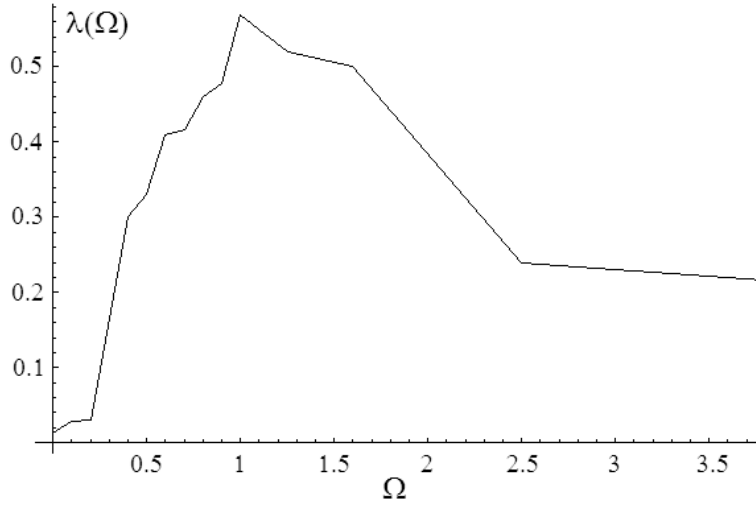

FIG. 4: Maximal Lyapunov exponent as a function of ratio between constants of interaction between atom and radiation field $(\Omega = g_2/g_1)$

of parameter $\Omega = 0$, maximal Lyapunov exponent is equal to zero $\lambda = 0$. This result is in a good agreement with the results obtained in [14].

Other important physical quantity characterizing classical dynamic stochasticity is the

fractional dimension of the systems phase space. For defining of fractional dimension we will use algorithm of P.Grassberger, I.Procaccia [19, 20, 21].

Let we have set of state vectors $(\chi_i, i = 1, 2, \cdots, N)$corresponding to the successive steps of numeric integration. In our case χ_i is the complete set of variables (2) with the values corresponding to the moments of time $t = t_i$. Then we can use numeric data for estimation of following expression

$$C(\varepsilon) = \lim_{N \to \infty} \frac{1}{N(N-1)} \sum_{i,j=1}^{N} \theta(\varepsilon - |\chi_i - \chi_j|)$$

where $\theta(x) = \begin{cases} 0 \ \ x < 0 \\ 1 \ \ x \geqslant 0 \end{cases}$ is the step function. According to the P.Grassberger, I.Procaccia [19, 20, 21] fractional dimension may be defined as

$$D = \lim_{\varepsilon \to 0} \frac{C(\varepsilon)}{\log(\varepsilon)}$$

Expected dependence of $C(\varepsilon)$ is ε^D. So, plot must be a line with angular coefficient D. Result of numeric calculations are presented on Fig.5.

II. QUANTUM MECHANICAL CONSIDERATION. MIXED STATE FORMATION.

On the basis of numeric results of previous section, we can conclude that dynamics is chaotic. In case of quantum consideration we neglect kinetic energy of atomic motion as compared to the level transition frequencies, and random nature of motion we shall try to take into account by considering $x(t)$ as time dependent random Wiener process. As a result in interaction representation we get:

$$i\frac{d|\psi(t)\rangle}{dt} = \widehat{V}|\psi(t)\rangle \tag{10}$$

where interaction operator is

$$\widehat{V} = \hbar \sum_{\alpha=1}^{2} g_\alpha(\widehat{b}_\alpha \widehat{R}_{3\alpha} + \widehat{b}_\alpha^+ \widehat{R}_{\alpha 3}) \cdot \cos k_{f_\alpha}\widehat{x} \tag{11}$$

Let present initial wave function as a direct product of atomic and field states

$$|\psi(0)\rangle = |\psi_{atom}\rangle \otimes |\psi_{field}\rangle$$
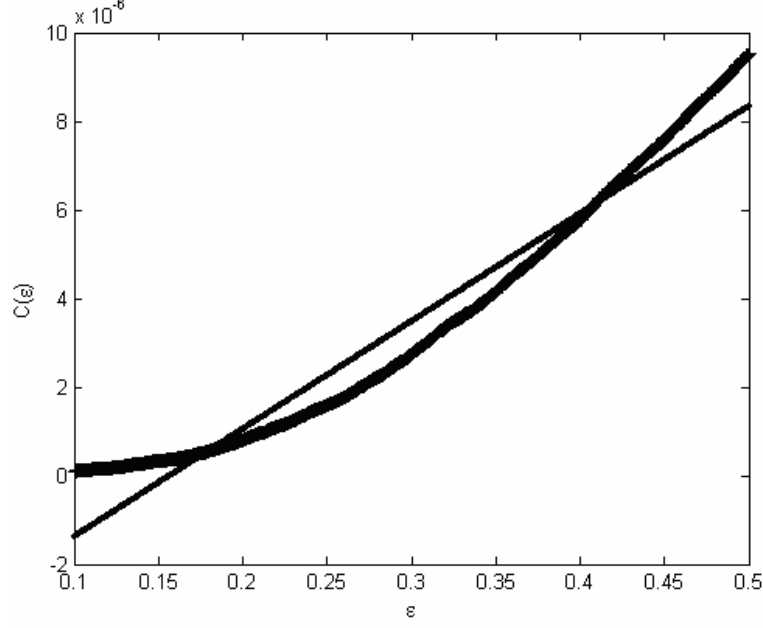
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FIG. 5: Dependence of $C(\varepsilon)$ on the values of ε , plotted by numerical integration of the set of equations (2) , for the following values of parameters $\alpha = 0.01$, $\Omega = 0.5$, $k = 3$, $M_1 = 1.6$, $M_2 = 1.7$. A solid line corresponds to least-squares approximation of the results of data processing. According to this plot, fractal dimension of the system $D = \frac{\ln C(\varepsilon_2) - \ln C(\varepsilon_1)}{\ln(\varepsilon_2) - \ln(\varepsilon_1)}$ is equal to $D \approx 2.44$

where

$$|\psi_{field}\rangle = \sum_{n_1=0}^{\infty} W_{n_1}|n_1\rangle + \sum_{n_2=0}^{\infty} W_{n_2}|n_2\rangle, |\psi_{atom}\rangle = C_{a_1}|a_1\rangle + C_{a_2}|a_2\rangle + C_{a_3}|a_3\rangle$$

$$|a_1\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, |a_2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, |a_3\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

As far as operator (11) can mix only following states $|a_3 \, n_1\rangle$, $|a_1 \, n_1+1\rangle$, $|a_3 \, n_2\rangle$, $|a_2 \, n_2+1\rangle$, solution of (10) we will search in the form

$$|\psi(t)\rangle = \sum_{a_j n_i} C_{a_j n_i}(t)|a_j n_i\rangle. \tag{12}$$

After substitution (12) in (10) and solving equations for $C_{a_j n_i}(t)$ we get

$$C_{a_3 n_1}(t) = \frac{C_1}{2} e^{-i\lambda_1 \int_0^t \cos k_{f_1} x(t')dt'} + \frac{C_2}{2} e^{i\lambda_1 \int_0^t \cos k_{f_1} x(t')dt'},$$
$$C_{a_1 n_1+1}(t) = \frac{C_1}{2} e^{-i\lambda_1 \int_0^t \cos k_{f_1} x(t')dt'} - \frac{C_2}{2} e^{i\lambda_1 \int_0^t \cos k_{f_1} x(t')dt'}. \tag{13}$$

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where $\lambda_1 = g_1\sqrt{n_1 + 1}$

$$C_1 = C_{a_3 n_1}(0) + C_{a_1 n_1+1}(0); \quad C_2 = C_{a_3 n_1}(0) - C_{a_1 n_1+1}(0);$$

$$
\begin{aligned}
C_{a_3 n_2}(t) &= \frac{C_3}{2} e^{-i\lambda_2 \int\limits_0^t \cos k_{f_2} x(t')dt'} + \frac{C_4}{2} e^{i\lambda_2 \int\limits_0^t \cos k_{f_2} x(t')dt'}, \\
C_{a_2 n_2+1}(t) &= \frac{C_3}{2} e^{-i\lambda_2 \int\limits_0^t \cos k_{f_2} x(t')dt'} - \frac{C_4}{2} e^{i\lambda_2 \int\limits_0^t \cos k_{f_2} x(t')dt'}.
\end{aligned}
\tag{14}
$$

Here $\lambda_2 = g_2\sqrt{n_2 + 1}$

$$C_3 = C_{a_3 n_2}(0) + C_{a_2 n_2+1}(0); C_4 = C_{a_3 n_2}(0) - C_{a_2 n_2+1}(0);$$

Density matrix of the system: field + atom is direct product of two density matrixes

$$\rho_{ijkl} = \rho_{atom} \otimes \rho_{field} = \overline{C_{a_i n_j} \cdot C^*_{a_k n_l}} \tag{15}$$

where $\overline{(\ldots)}$ means averaging.

Sign of mixed state formation is zeroing of non-diagonal matrix elements of density matrix (15), [22, 23]. This process is non-reversible because, along with zeroing of non-diagonal matrix elements, loss of information about wave function's phase factor takes place. So, to prove irreversibility of dynamics, it is sufficient to show zeroing of non-diagonal matrix elements.

Substituting (13), (14) into (15), it is easy to see that non-diagonal matrix elements contain terms like

$$C_{a_i n_j}(0) C^*_{a_k n_l}(0) \exp\left[i \int\limits_0^t \omega^\pm_{1,2}(t')dt'\right] \tag{16}$$

where $\omega^\pm_{1,2}(t') = g_1 \cos(k_{f_1} x(t')) \pm g_2 \cos(k_{f_2} x(t'))$. Due to the random nature of $x(t')$, below we will consider $\omega^\pm_{1,2}$ as a random process and for simplicity omit indices. It is clear that exponent in (16) may be considered as a functional of the random function $\omega(t)$

$$Q[\omega] = \exp\left[i \int\limits_0^t \omega(t')dt'\right]. \tag{17}$$

So, to get non-diagonal matrix elements of the density matrix (15), we need to do statistical averaging with respect to the all possible realizations of random function $\omega(t)$

$$\rho_{\substack{ijkl, \\ i\neq k, j\neq l}} \approx C_{a_i n_j}(0) C^*_{a_k n_l}(0) \langle Q[\omega[t]] \rangle \tag{18}$$

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where $\langle(\dots)\rangle$ means statistical average. Average values of the functional (18) may be calculated by doing following continual integral

$$Q[\omega] = \exp\left[i\int_0^t \omega(t')dt'\right] = \lim_{\substack{N\to\infty \\ \Delta t_k \to 0}} \int d\omega_N \dots d\omega_1 \exp\left[i\sum_{k=1}^n \omega_k \Delta t_k\right] P_N[\omega] \qquad (19)$$

where $\Delta t_k = t^{(k)} - t^{(k-1)}$, $t^{(0)} = 0$, $t^{(N)} = 1$, $P_N[\omega]$ is the multi-dimensional normal distribution function given by

$$P_N[\omega] = (2\pi)^{-N}\int d\lambda_1 \dots d\lambda_N \exp\left[-i\sum_k \lambda_k \omega_k\right]\exp\left[-\frac{1}{2}\sum_{k,k'} C_{kk'}\lambda_k\lambda_{k'}\right] \qquad (20)$$

where λ_k are the distribution parameters and $C_{kk'}$ is the covariation matrix [24]

After substituting (20) in (19) and doing integration we obtain

$$\int d\omega_1 \dots d\omega_N \exp[i\sum_{k=1}^N \omega_k \Delta t_k]P_N[\omega] =$$

$$= \int d\lambda_1 \dots d\lambda_N \exp\left[-\frac{1}{2}\sum_{k,k'} C_{k,k'}\lambda_k\lambda_{k'}\right] \times \prod_{k=1}^N \left\{\frac{1}{2\pi}\int \exp[i\omega_k(\Delta t_k - \lambda_k)]d\omega_k\right\} = \qquad (21)$$

$$= \int d\lambda_1 \dots d\lambda_N \delta(\lambda_1 - \Delta t_1)\delta(\lambda_2 - \Delta t_2)\dots\delta(\lambda_N - \Delta t_N) \times \exp\left[-\frac{1}{2}\sum_{k,k'} \Delta t_k \Delta t_{k'}\right]$$

So, for average value of the functional we get

$$\langle Q[\omega]\rangle = \lim_{\substack{N\to\infty \\ \Delta t_k \to 0}} \exp\left[-\frac{1}{2}\sum_{k,k'} c(t^{(k)}, t^{(k')})\Delta t_k \Delta t_{k'}\right] = \exp\left[-\frac{1}{2}\int_0^t dt' \int_0^{t''} c(t', t'')\right] \qquad (22)$$

For normal random process $c(t', t'') = c(t' - t'')$. Then making transform to the new variables $t' - t'' = \tau$, $t' + t'' = \xi$ and doing integration in (22) over variable ξ we get

$$\langle Q[\omega]\rangle = \exp\left[-\frac{1}{2}t\int_{-t}^t d\tau c(\tau)\right]. \qquad (23)$$

After assuming that correlation function $c(\tau) = \langle\omega(t+\tau)\omega(t)\rangle$ has the Gaussian form $c(\tau) = e^{-\alpha_0\tau^2}$, from (23) we get

$$\langle Q[\omega]\rangle \approx \exp\left[-\sqrt{\frac{\pi}{\alpha_0}}\frac{t}{2}\cdot Erf[t\sqrt{\alpha_0}]\right] \qquad (24)$$

where $Erf(\dots)$ is the error function [24].

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So, for non-diagonal matrix elements we have

$$\rho_{\substack{ijkl \\ i\neq k, j\neq l}} \approx C_{a_i n_j}(0) C^*_{a_k n_l}(0) \exp\left[-\sqrt{\frac{\pi}{\alpha_0}}\frac{t}{2} \cdot Erf[t\sqrt{\alpha_0}]\right]. \tag{25}$$

From (25) it is clear that after the lapse of time $t > \sqrt{\alpha_0/\pi}$ zeroing of non-diagonal matrix elements and formation of mixed state happens.

III. KINETIC DESCRIPTION

In previous section we investigated mechanism of mixed state formation. After mixed state is formed, quantum-mechanical consideration loses meaning and there is a need to use kinetic description. Kinetic equation for chaotic quantum-mechanical system first time was obtained in [16]. But this study was done in the semi-classical domain. Namely, in [16] zeroing of non-diagonal part of density matrix was proved by use of semi-classical approximation. Our purpose is to do the same in exceptionally quantum domain, without application to the semi-classical methods. As the zeroing of non-diagonal part is already shown, problem is partly solved.

It's clear that system atom+ field is adiabatically isolated. So entropy product is zero. But by considering field as a thermostat, we can calculate entropy growth for atomic sub-system.If $\Delta S > 0$, then process in non-reversible.

As we have already mentioned at $t = 0$, atom is not connected with light and state vector $|\psi(t = 0)\rangle$ is the direct product of two states

$$|\psi(0)\rangle = |\psi_{atom}\rangle \otimes |\psi_{field}\rangle \tag{26}$$

where

$$|\psi_{field}\rangle = \sum_{n_1=0}^{\infty} W_{n_1}|n_1\rangle + \sum_{n_2=0}^{\infty} W_{n_2}|n_2\rangle, \quad |\psi_{atom}\rangle = C_{a_1}|a_1\rangle + C_{a_2}|a_2\rangle + C_{a_3}|a_3\rangle \tag{27}$$

Let us note that interaction (11) mixes only states

$$|a_3\ n_1\rangle, \quad |a_1\ n_1 + 1\rangle, \quad |a_3\ n_2\rangle, \quad |a_2\ n_2 + 1\rangle.$$

Then comparing (26) with (12) one can see that initial conditions in (13), (14) are of form

$$
\begin{aligned}
C_{a_3 n_1}(0) &= W_{n_1} C_{a_3}, \\
C_{a_1 n_1+1}(0) &= W_{n_1+1} C_{a_1}, \\
C_{a_3 n_2}(0) &= W_{n_2} C_{a_3}, \\
C_{a_2 n_2+1}(0) &= W_{n_2+1} C_{a_2}.
\end{aligned}
\tag{28}
$$

Let us remind that inversion of atomic populations I is a value of interest, since it can be measured on the experiment. Inversion is defined as difference between level populations

$$
\begin{aligned}
I_{a_3 a_2}(t) &= W(t, |a_3\rangle) - W(t, |a_2\rangle), \\
I_{a_3 a_1}(t) &= W(t, |a_3\rangle) - W(t, |a_1\rangle),
\end{aligned}
\tag{29}
$$

where

$$
W(t, |a\rangle) = \sum_{n=0}^{\infty} |C_{an}(t)|^2.
\tag{30}
$$

After substituting (13), (14) in (30), and using initial conditions (28) we get

$$
\begin{aligned}
W(t, |a_3\rangle) &= \tfrac{1}{4} \sum_{n_2=0}^{\infty} \left| (W_{n_2} C_{a_3} + W_{n_2+1} C_{a_2}) Q_{n_2}^*[\omega_2] + (W_{n_2} C_{a_3} - W_{n_2+1} C_{a_2}) Q_{n_2}[\omega_2] \right|^2 + \\
&\quad + \tfrac{1}{4} \sum_{n_1=0}^{\infty} \left| (W_{n_1} C_{a_3} + W_{n_1+1} C_{a_1}) Q_{n_1}^*[\omega_1] + (W_{n_1} C_{a_3} - W_{n_1+1} C_{a_1}) Q_{n_1}[\omega_1] \right|^2, \\
W(t, |a_2\rangle) &= \tfrac{1}{4} \sum_{n_2=0}^{\infty} \left| (W_{n_2} C_{a_3} + W_{n_2+1} C_{a_2}) Q_{n_2}^*[\omega_2] - (W_{n_2} C_{a_3} - W_{n_2+1} C_{a_2}) Q_{n_2}[\omega_2] \right|^2, \\
W(t, |a_1\rangle) &= \tfrac{1}{4} \sum_{n_1=0}^{\infty} \left| (W_{n_1} C_{a_3} + W_{n_1+1} C_{a_1}) Q_{n_1}^*[\omega_1] - (W_{n_1} C_{a_3} - W_{n_1+1} C_{a_1}) Q_{n_1}[\omega_1] \right|^2.
\end{aligned}
\tag{31}
$$

To get we also have introduced following notation for unitary functional

$$
Q_{n_{1,2}}[\omega_{1,2}] = \exp \left[-i g_{1,2} \sqrt{n_{1,2}+1} \int_0^t \cos(k_{f_{1,2}} x(t')) dt' \right],
\tag{32}
$$

$$
Q_{n_{1,2}}^*[\omega_{1,2}] = Q_{n_{1,2}}^{-1}[\omega_{1,2}], \quad Q_{n_{1,2}}[\omega_{1,2}] Q_{n_{1,2}}^{-1}[\omega_{1,2}] = 1.
\tag{33}
$$

By using of (31) it is easy to define inversion values for any moment of time. But before doing this let us mention one interesting fact. In ordinary JC model which for zero detuning is not characterized by chaos, inversion is characterized by periodical revivals in time. See for example [25]. So absence of periodical revivals maybe useful for experimental observation of quantum chaos. After using (32), (33) from (31) we get

$$
\langle W(t, |a_2\rangle) \rangle = \sum_{n_2=0}^{\infty} \left[\frac{1}{2} (W_{n_2}^2 C_{a_3}^2 + W_{n_2+1}^2 C_{a_2}^2) - \frac{1}{4} (W_{n_2}^2 C_{a_3}^2 - W_{n_2+1}^2 C_{a_2}^2)(\langle Q_{n_2}^2[\omega_2] \rangle + \langle Q_{n_2}^{-2}[\omega_2] \rangle) \right]
\tag{34}
$$

$$\langle W(t,|a_1\rangle)\rangle = \sum_{n_1=0}^{\infty} [\frac{1}{2}(W_{n_1}^2 C_{a_3}^2 + W_{n_1+1}^2 C_{a_1}^2) - \frac{1}{4}(W_{n_1}^2 C_{a_3}^2 - W_{n_1+1}^2 C_{a_2}^2)(\langle Q_{n_1}^2[\omega_1]\rangle + \langle Q_{n_1}^{-2}[\omega_1]\rangle)]$$

(35)

where $\langle \dots \rangle$ again means statistical average. Then taking into account(24)

from (34), (35) we get

$$\begin{aligned}
\langle W(t,|a_2\rangle)\rangle &= \sum_{n_2=0}^{\infty} [\tfrac{1}{2}(W_{n_2}^2 C_{a_3}^2 + W_{n_2+1}^2 C_{a_2}^2)], \\
\langle W(t,|a_1\rangle)\rangle &= \sum_{n_1=0}^{\infty} [\tfrac{1}{2}(W_{n_1}^2 C_{a_3}^2 + W_{n_1+1}^2 C_{a_1}^2)], \\
\langle W(t,|a_3\rangle)\rangle &= \langle W(t,|a_1\rangle)\rangle + \langle W(t,|a_2\rangle)\rangle .
\end{aligned}$$

(36)

At last let us remind that $W_{n_{1,2}}^2$ describes field states and obeys to Poisson distribution [26]

$$W_{n_{1,2}}^2 = \frac{\bar{n}_{1,2}^{n_{1,2}} \exp(-\bar{n}_{1,2})}{n_{1,2}!},$$

(37)

and C_{a_1}, C_{a_2}, C_{a_3} are probabilities of level occupations.

Let us assume that, at the initial moment of time, atom is in the lowest state

$$\rho_{a_1}(t=0) = \left|C_{a_1 n_1+1(t=0)}\right|^2 = 1, \quad \rho_{a_2}(t=0) = \rho_{a_3}(t=0) = 0.$$

(38)

Then initial values of entropy is zero

$$S(t=0) = \sum_{a_i=1}^{3} \rho_{a_i}(t=0) \ln(\rho_{a_i}(t=0)) = 0,$$

(39)

so system is in pure quantum-mechanical state.

After the lapse of time $t = t_0$ more than time of inter level transition $t_0 \sim 1/g_\alpha$, system may perform multiple transitions between levels. That is why probability to find system in other states will be nonzero:

$$C_{a_1} \neq 0, \quad C_{a_2} \neq 0, \quad C_{a_3} \neq 0, \quad t > t_0.$$

(40)

Despite of this fact to talk about probability of population of different states is early yet. The point is that in time interval:

$$t_0 < t < \sqrt{\frac{\alpha_0}{\pi}}$$

(41)

interferential terms are nonzero. Therefore the state of the system will be pure one. But unlike of the initial state (38),which is simple state, the state of the system in time interval (41) is superposition one.

Superposition state is pure quantum mechanical state and only after zeroing of interferential terms in (18) superposition state passes to mixed one. Such a transition occurs in times:

$$t > \sqrt{\frac{\alpha_0}{\pi}} \qquad (42)$$

But in time interval (41) while the system is in pure superposition state, from the symmetry point of view, it is clear that the coefficient values(40) have to satisfy the following relation:

$$C_{a_1}\left(t_0 < t < \sqrt{\frac{\alpha_0}{\pi}}\right) \sim C_{a_2}\left(t_0 < t < \sqrt{\frac{\alpha_0}{\pi}}\right) \sim C_{a_3}\left(t_0 < t < \sqrt{\frac{\alpha_0}{\pi}}\right) \sim C. \qquad (43)$$

Before proceed to the more exact determination of the quantities (36), we note one important fact. Values of the parameters (28) C_{a_1}, C_{a_2}, C_{a_3}, taken at the initial moment of time $t = 0$, define level populations. At the moment of time $t = 0$ atom and cavity field are not connected with each other (26). Therefore the following relation should be hold:

$$C_{a_1}^2 + C_{a_2}^2 + C_{a_3}^2 = 1. \qquad (44)$$

After the laps of time interaction (11) will mix atom and field's states and due to the fact that (41) does not correspond to the normalization condition of complete wave function, it will not be valid henceforth. Considering of level populations again is possible only after the time interval (42). But in this case level populations are defined by the quantities (36), (28) and not by the coefficients (69). fficients (69). For determination of level's populations we shall use (36),(43) and take into account that fields states $W_{n_{1,2}}$ obeys Poisson distribution (37). As a result we obtain

Then taking into account (36)as a result we get

$$\rho_{a1} = \frac{1}{4}, \quad \rho_{a2} = \frac{1}{4}, \quad \rho_{a3} = \frac{2}{4}, \qquad (45)$$

Result is very interesting. Most populated is high exited level ρ_{a_3}. This means that we have stochastic absorption of field energy. For entropy growth we have

$$\Delta S = \frac{1}{2}(\ln 4 + \ln 2) > 0 \qquad (46)$$

So, process is non-reversible.

IV. **CONCLUSIONS**

Let sum up and analyze the results obtained in conclusion.

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The aim of this work was to study generalized JC model being subject to resonator field. Interest to such a systems is caused by the fact that they are the most perspective to be used in quantum computer. The question that came up is the following: by how much will be state of the system controllable and dynamics reversible? We have considered the most general case, when interaction of the system with field depends on coordinate of the system inside resonator.

Contrary to generally accepted opinion, it has turned out that the absence of detuning, between resonator field and frequency of the inter-level transitions, does not guarantee reversibility of the system's state. During evolution in time the system executes irreversible transition from pure quantum-mechanical state to mixed one. At the same time, the time needed for formation of mixed state $t > \sqrt{\alpha_0/\pi}$ is determined completely by the autocorrelation function of random variable $\omega(t)$. Randomsity of this variable in it's turn is connected to the chaotic motion of atoms inside of cavity.

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